Advanced Automatic Control MDP 444

"If you have a smart project, you can say "I'm an engineer""

Lecture 6

Staff boarder

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Advanced Automatic Control
MDP 444

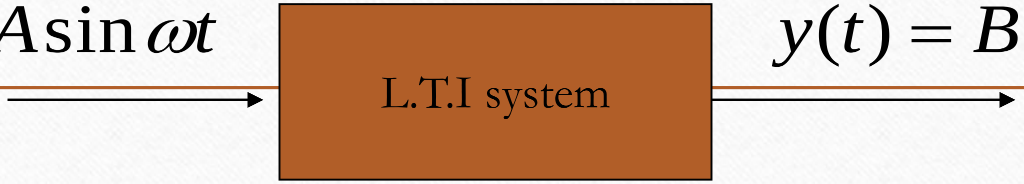
- **Lecture aims:**
  - Understand the powerful concept of frequency response and its role in control system design
  - Understand performance specifications in the frequency domain and relative stability based on gain and phase margins

Frequency Response

- we examined the use of test signals such as a step and a ramp signal.
- In this lecture, we consider the steady-state response of a system to a **sinusoidal input** test signal.
- We will examine the transfer function $G(s)$ when $s = j\omega$ and develop methods for graphically displaying the complex number $G(j\omega)$ as ω varies. The **Bode plot** is one of the most powerful graphical tools for analyzing and designing control systems.
- The **frequency response** of a system is defined as the steady-state response of the system to a sinusoidal input signal. The **sinusoid is a unique** input signal, and the resulting output signal for a linear system, as well as signals throughout the system, is sinusoidal in the steady state; it differs from the input waveform only in amplitude and phase angle.

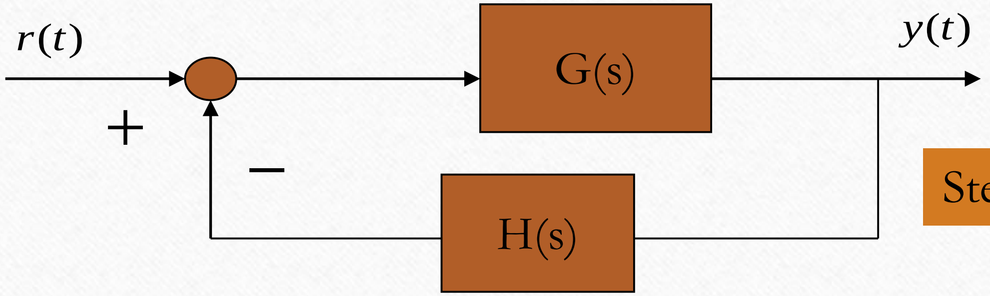
Frequency Response

$$r(t) = A\sin\omega t \qquad\qquad y(t) = B\sin(\omega t + \phi)$$



Magnitude: $\dfrac{B}{A}$

Phase: ϕ



Steady state response

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \qquad\qquad s = \sigma + j\omega \Rightarrow s = j\omega$$

Magnitude: $\dfrac{|G(j\omega)|}{|1+G(j\omega)H(j\omega)|}$

Phase: $\dfrac{\angle G(j\omega)}{\angle[1+G(j\omega)H(j\omega)]}$

Poles and Zeros and Transfer Functions

Transfer Function: A transfer function is defined as the ratio of the Laplace transform of the output to the input with all initial conditions equal to zero. Transfer functions are defined only for linear time invariant systems.

Considerations: Transfer functions can usually be expressed as the ratio of two polynomials in the complex variable, s.

Factorization: A transfer function can be factored into the following form.

$$G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

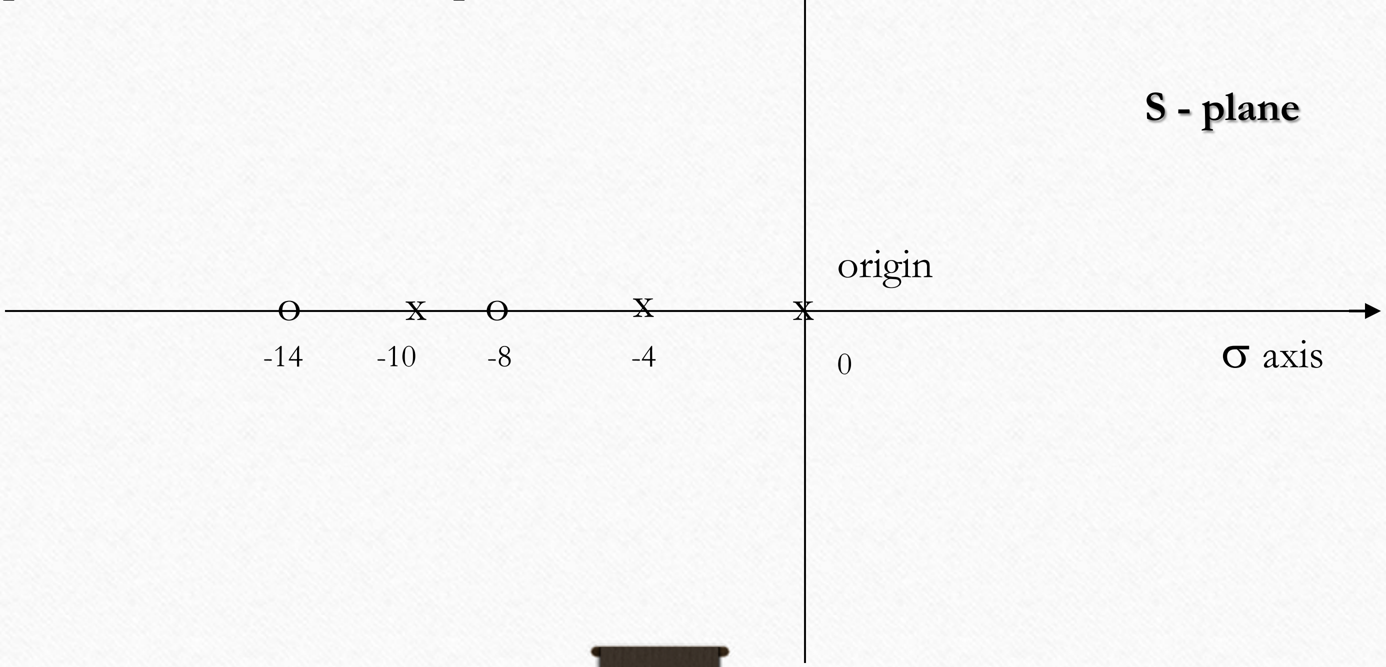
The roots of the numerator polynomial are called <u>zeros.</u>

The roots of the denominator polynomial are called <u>poles</u>.

Poles, Zeros and the S-Plane

An Example: You are given the following transfer function. Show the poles and zeros in the s-plane.

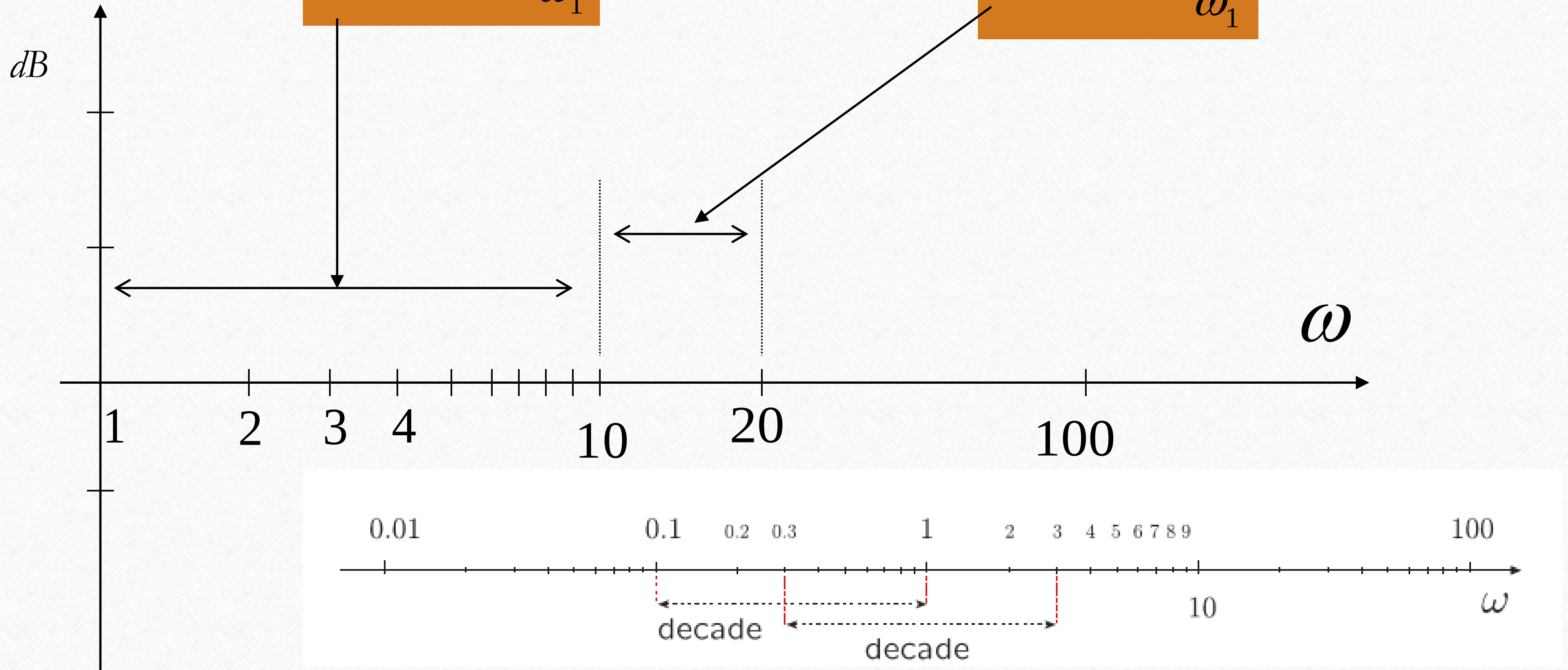
$$G(s) = \frac{(s+8)(s+14)}{s(s+4)(s+10)}$$



Logarithmic coordinate

Decade : $dec = \log_{10} \frac{\omega_2}{\omega_1}$

Octave : $oct = \log_2 \frac{\omega_2}{\omega_1}$



we use a logarithmic (log10) scale for the abscissa (angular frequency ω)

Logarithmic coordinate

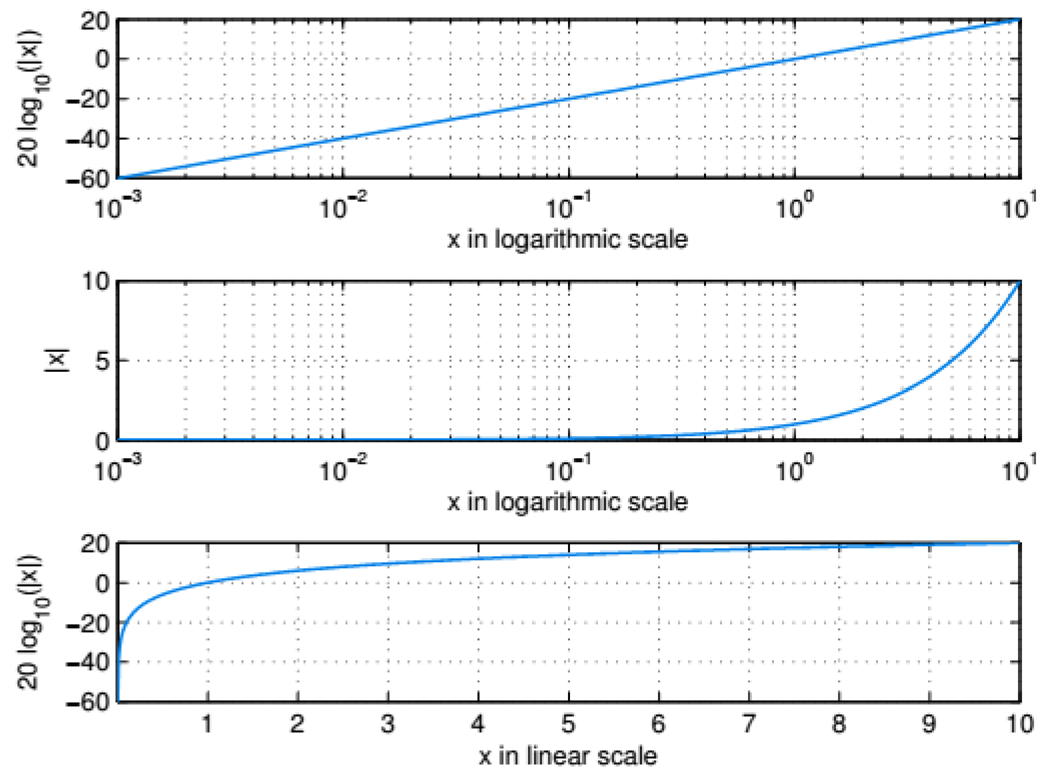
$\log_{10}(\omega)$ becomes a straight line

if ω is in a logarithmic scale

↓

very useful when we add

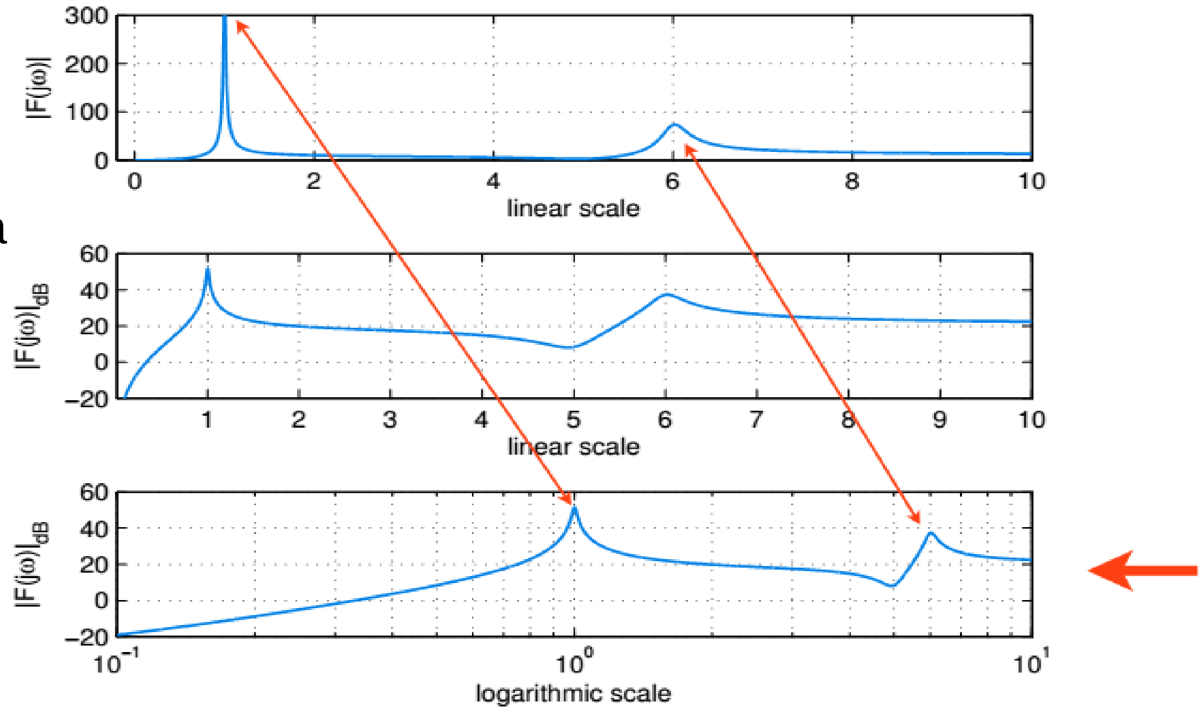
different contributions



Logarithmic coordinate

advantages

- quantities can vary in large range (both ω and magnitude)
- easy to build the magnitude plot in dB of a frequency response given in its Bode canonical form from the magnitudes of the single terms
- easy to represent series of systems
- same data different scales for abscissa and ordinates



Poles, Zeros and Bode Plots

Characterization:

Considering the transfer function of the previous slide. We note that we have 4 different types of terms in the previous general form:

$$G(jw) = \frac{K_B(jw/z+1)}{(jw)(jw/p+1)}$$

These are:

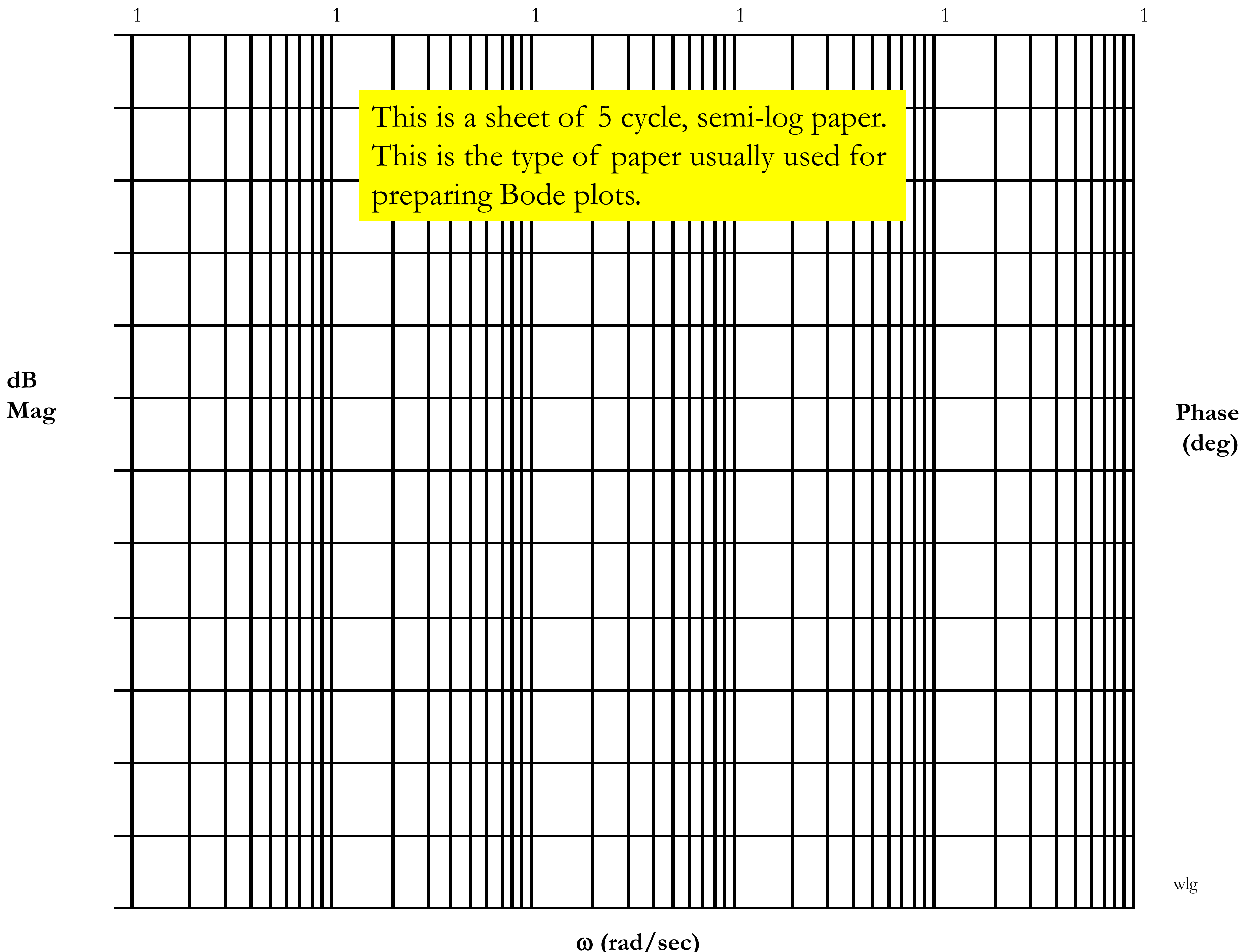
$$K_B, \quad \frac{1}{s}, \quad \frac{1}{(s/p\ + 1)}, \quad (s/z+1)$$

Expressing in dB:

Given the tranfer function:

$$20\log|G(jw| = 20\log K_B + 20\log|(jw/z+1)| - 20\log|jw| - 20\log|jw/p+1|$$

This is a sheet of 5 cycle, semi-log paper. This is the type of paper usually used for preparing Bode plots.



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Poles, Zeros and Bode Plots

- A **Bode plot** is a (semilog) plot of the transfer function **magnitude** and **phase** angle as a function of frequency
- The gain magnitude is many times expressed in terms of decibels (dB)

$$dB = 20 \log_{10} A$$

where A is the amplitude or gain

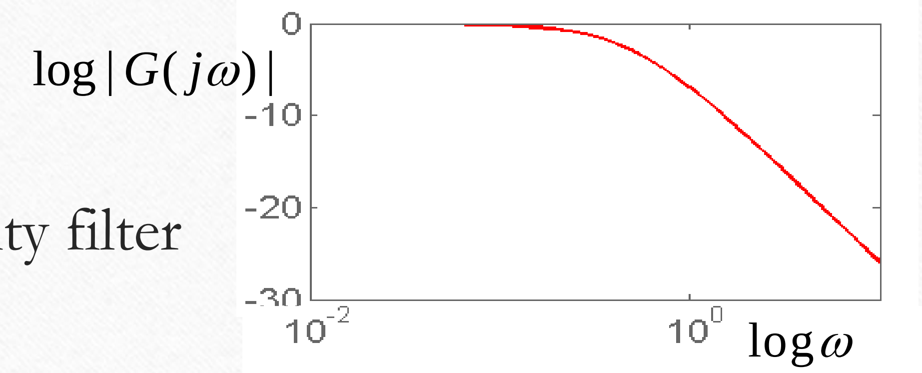
  - a *decade* is defined as any 10-to-1 frequency range
  - an *octave* is any 2-to-1 frequency range

$$20 \text{ dB/decade} = 6 \text{ dB/octave}$$

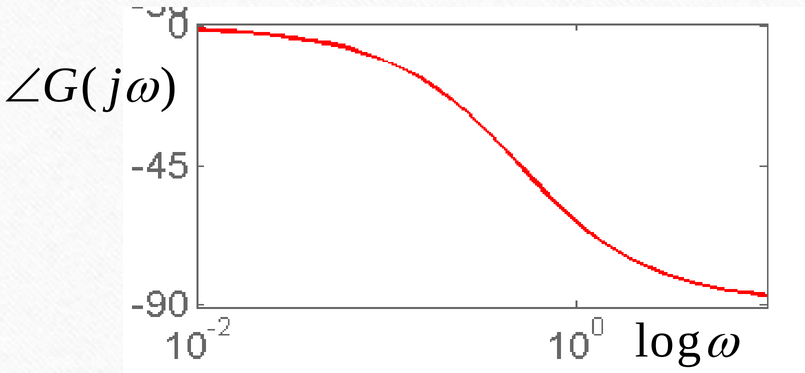
Poles, Zeros and Bode Plots

- A **Bode Plot** for a system is simply plots of log magnitude and phase against log frequency
- Both the log magnitude and phase effects are now **additive**
- Widely used for **analysis and design** of **filters** and **controllers**
- **Example**
- Low pass, unity filter

Log mag v log freq



Phase v log freq



Poles, Zeros and Bode Plots

$\|F(j\omega)\|_{dB}$	magnitude in dB of the frequency response as a function of the angular frequency ω with logarithmic scale for ω
$\angle F(j\omega)$	angle or phase of the frequency response as a function of the angular frequency ω with logarithmic scale for ω

we need to find the magnitude (in dB) and phase for the 4 elementary factors

1. Constant K (generalized gain)
2. Monomial $j\omega$ (zero or pole in $s = 0$)
3. Binomial $1 + j\omega\tau$ (non-zero real zero or pole)
4. Trinomial $1 + 2\zeta(j\omega)/\omega_n + (j\omega)^2/\omega_n^2$ (complex conjugate pairs of zeros or poles)

$20\log K_B$

$20\log|(jw/z + 1)|$

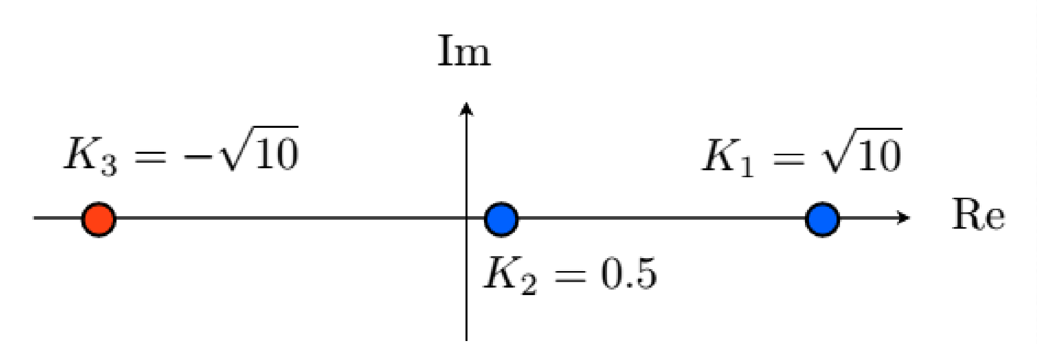
$-20\log|jw|$

$-20\log|(jw/p + 1)|$

Poles, Zeros and Bode Plots

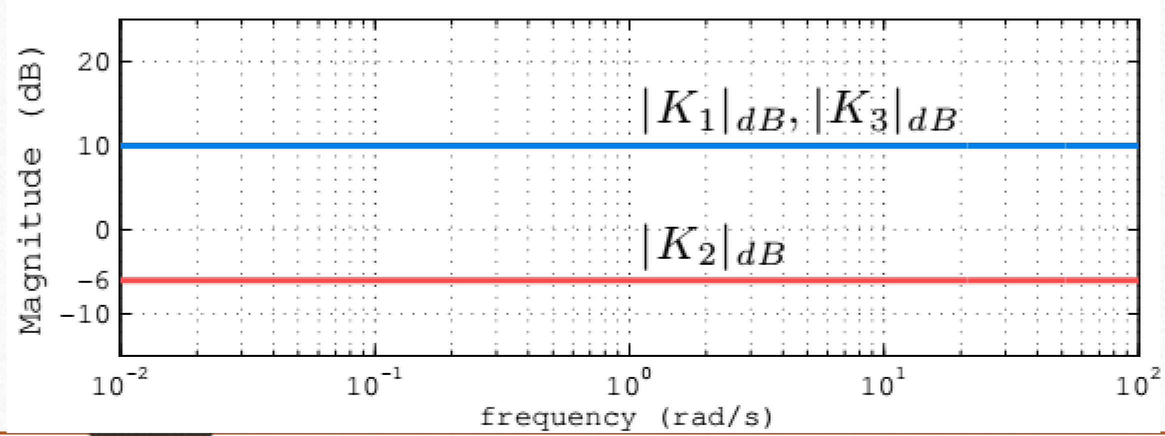
1. **Constant K (generalized gain)**

- The gain term, $20\log K_B$, is just so many dB and this is a straight line on Bode paper, independent of omega (radian frequency).



magnitude

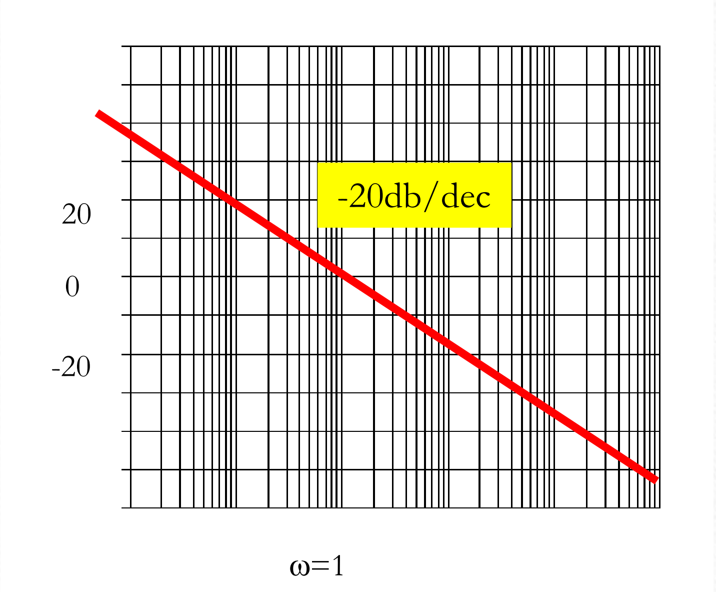
$$20 \log_{10} |K|$$



Poles, Zeros and Bode Plots

The term, - 20log|jw| = - 20logw, when plotted on semi-log paper is a straight line sloping at 20dB/decade.

It has a magnitude of 0 at w = 1.



Poles, Zeros and Bode Plots

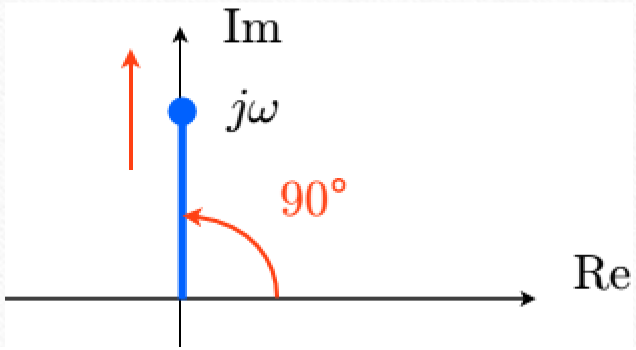
2. Monomial jω (Zeros) s^p

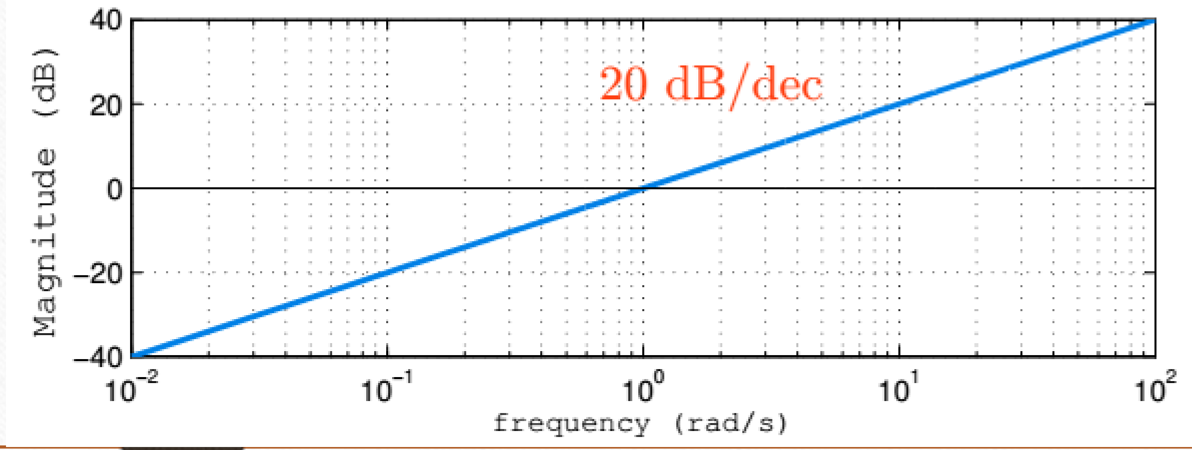
Magnitude:

$$\left|(j\omega)^p\right|_{dB} = 20p\log\omega(dB)$$

Phase:

$$\angle(j\omega)^p = (90^o)\times p$$





Poles, Zeros and Bode Plots

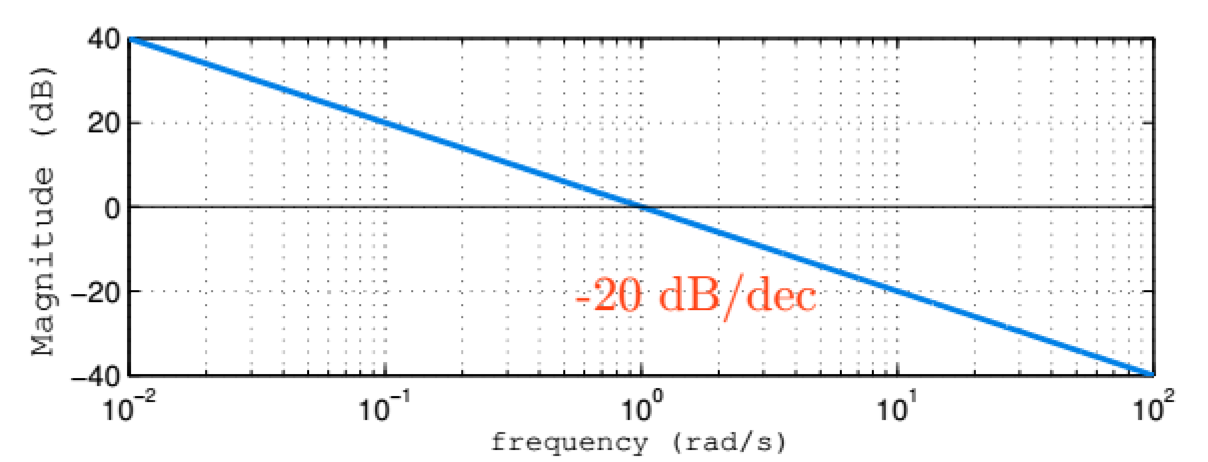
2. Monomial jω (Poles) $\dfrac{1}{s^{p}}$

Magnitude:

$$\left|\frac{1}{(j\omega)^{p}}\right|_{dB} = -20p\log\omega (dB)$$

Phase:

$$\angle\frac{1}{(j\omega)^{p}} = (-90^{o})\times p$$



Poles, Zeros and Bode Plots

3. Binomial 1 + jωτ (non-zero real zero or pole)

Magnitude approximation wrt the cutoff frequency $1/|\tau|$ (corner frequency)

$$|1 + j\omega\tau|_{dB} = 20\log_{10}\sqrt{1+\omega^2\tau^2}$$

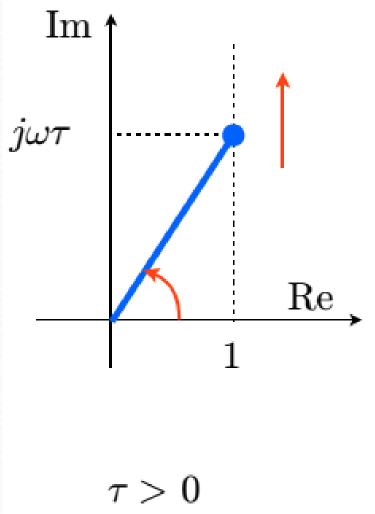
$$|1 + j\omega\tau|_{dB} \approx \begin{cases} 0\,dB & \text{if} \quad \omega \ll 1/|\tau| \\ 20\log_{10}\omega + 20\log_{10}|\tau| & \text{if} \quad \omega \gg 1/|\tau| \end{cases}$$

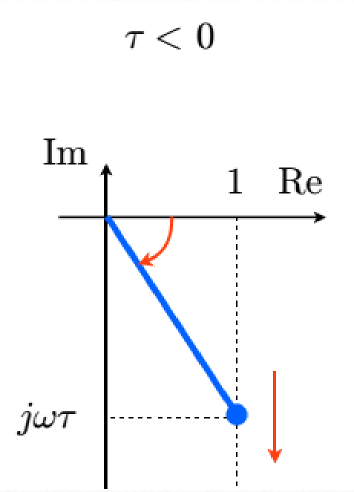
two half-lines approximation: 0 dB until the cutoff frequency, + 20dB/decade after

Poles, Zeros and Bode Plots

3. Binomial 1 + jωτ (non-zero real zero or pole)

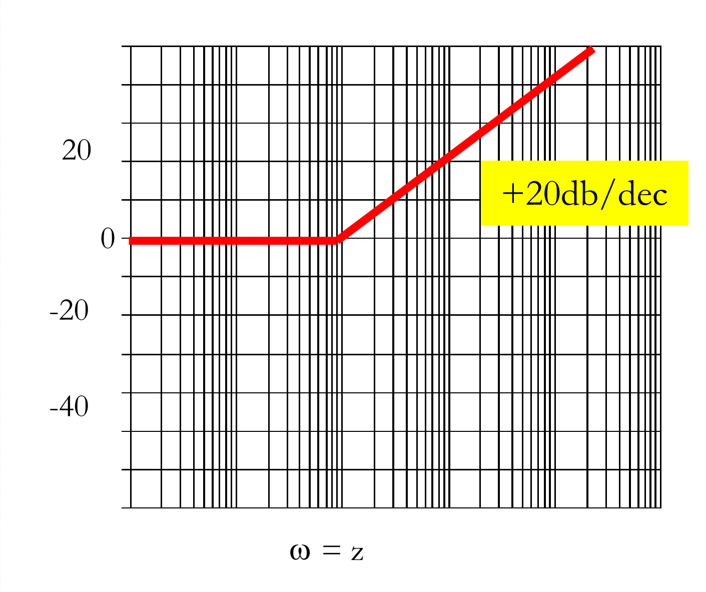
phase depends on the sign of τ





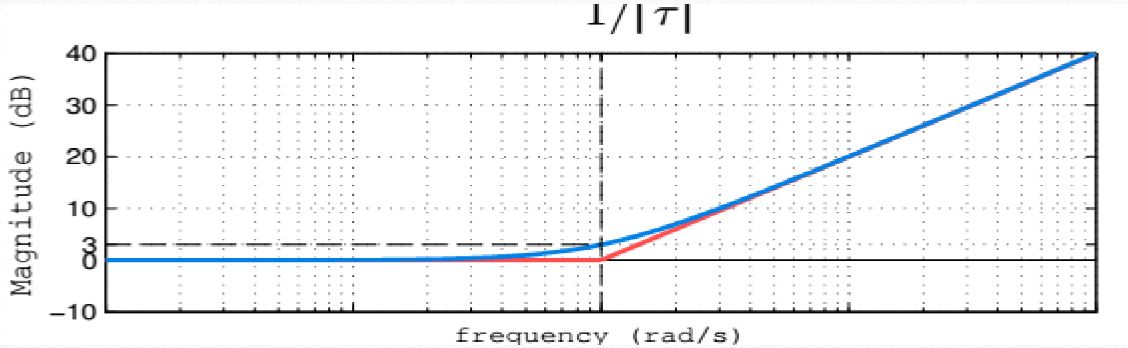
Poles, Zeros and Bode Plots

When we have a term of $20\log|(jw/z + 1)|$ we approximate it be a straight line of slop 0 dB/dec when **w** < **z**. We approximate it as $20\log(w/z)$ when **w** > **z**, which is a straight line on Bode paper with a slope of + 20dB/dec.

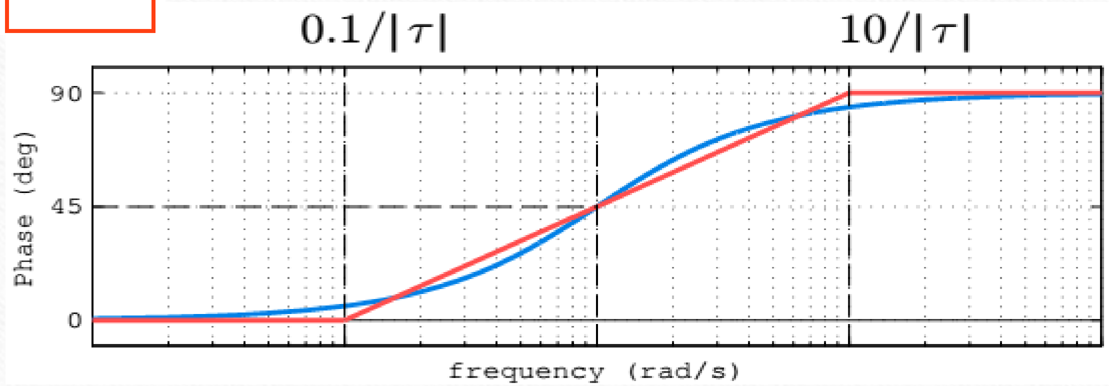


Poles, Zeros and Bode Plots

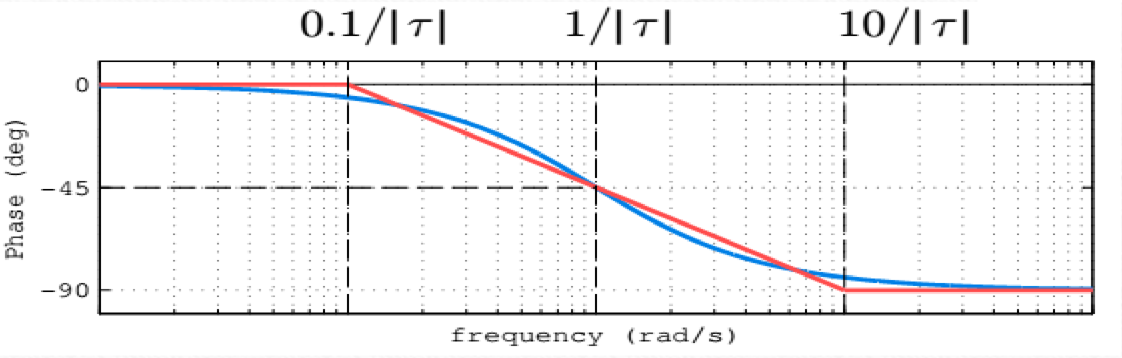
3. Binomial $1 + j\omega\tau$



$\tau > 0$



$\tau < 0$



Poles, Zeros and Bode Plots

Magnitude:

$$\left|(1+j\frac{\omega}{a})\right|_{dB} = 20\log\sqrt{1+(\frac{\omega}{a})^2}$$

$$=10\log[1+(\frac{\omega}{a})^2]$$

$$\omega \prec\prec a \Rightarrow \frac{\omega}{a} \approx 0 \Rightarrow dB = 10\log 1 = 0$$

$$\omega \succ\succ a \Rightarrow 1+j\frac{\omega}{a} \approx \frac{\omega}{a} \Rightarrow dB \approx 20\log\frac{\omega}{a}$$

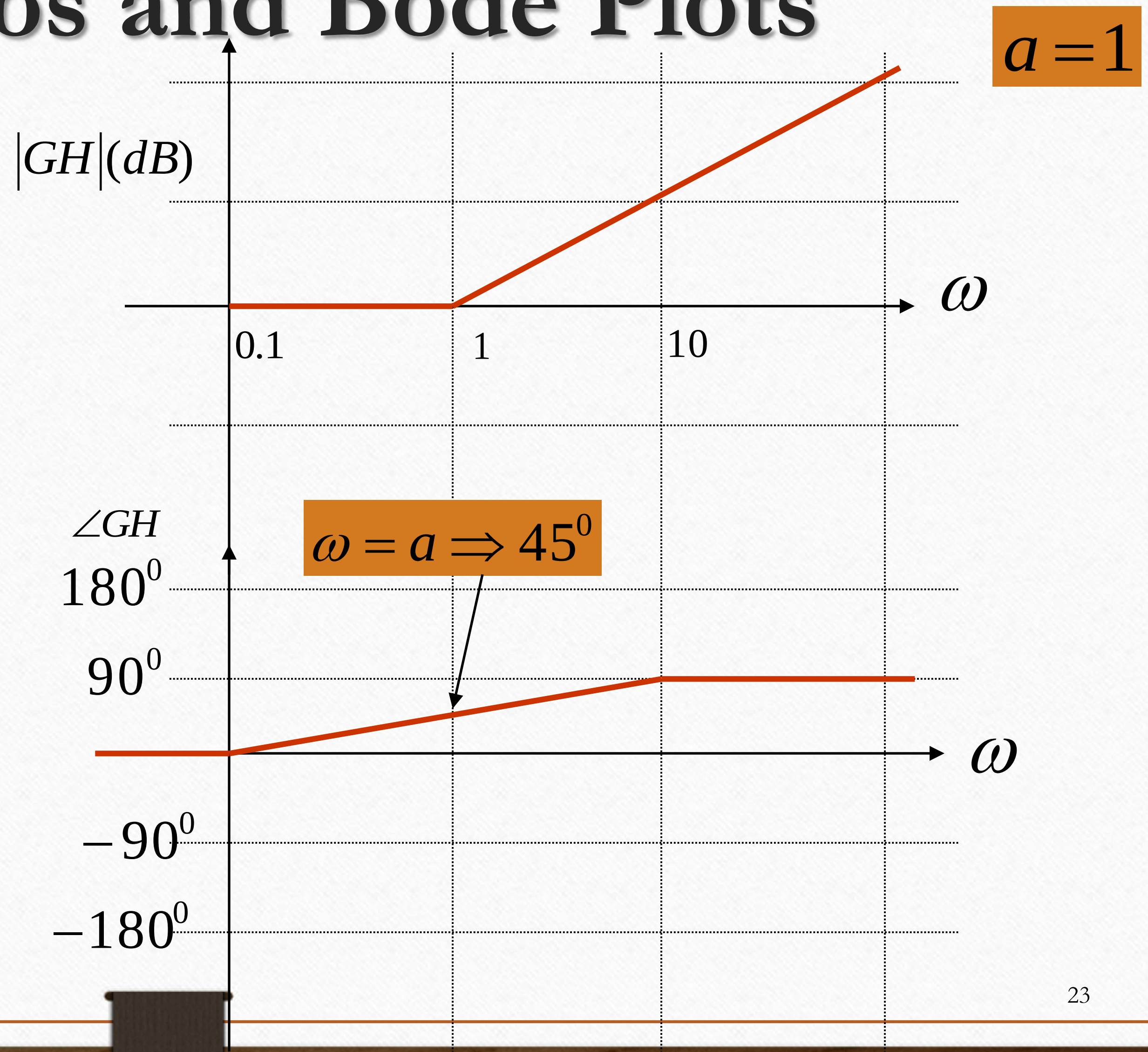
$$dB = 20\log\omega - 20\log a$$

$$\omega = a \Rightarrow 1+j1 \Rightarrow dB = 10\log 2 = 3.01$$

Phase: $$\angle(1+j\frac{\omega}{a}) = \tan^{-1}\frac{\omega}{a}$$

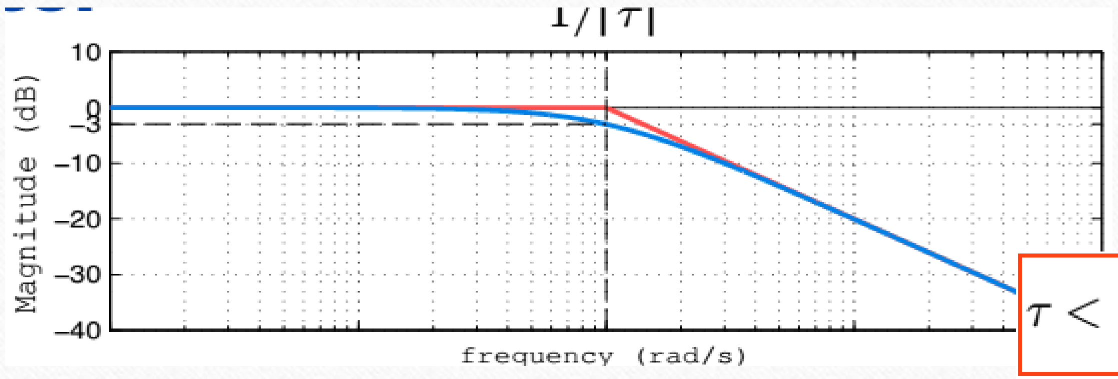
$$\omega \prec\prec a \Rightarrow \frac{\omega}{a} \approx 0 \Rightarrow \angle GH \approx \tan^{-1} 0 = 0^o$$

$$\omega \succ\succ a \Rightarrow \frac{\omega}{a} \approx \infty \Rightarrow \angle GH \approx \tan^{-1}\infty = 90^o$$



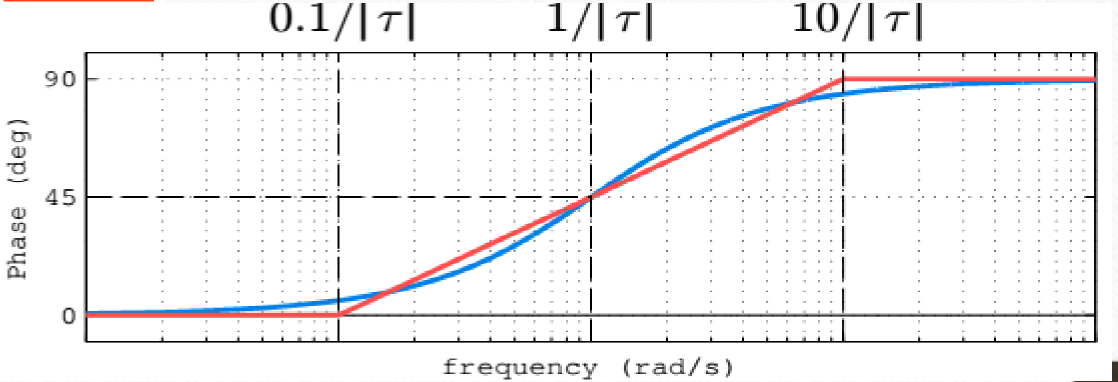
Poles, Zeros and Bode Plots

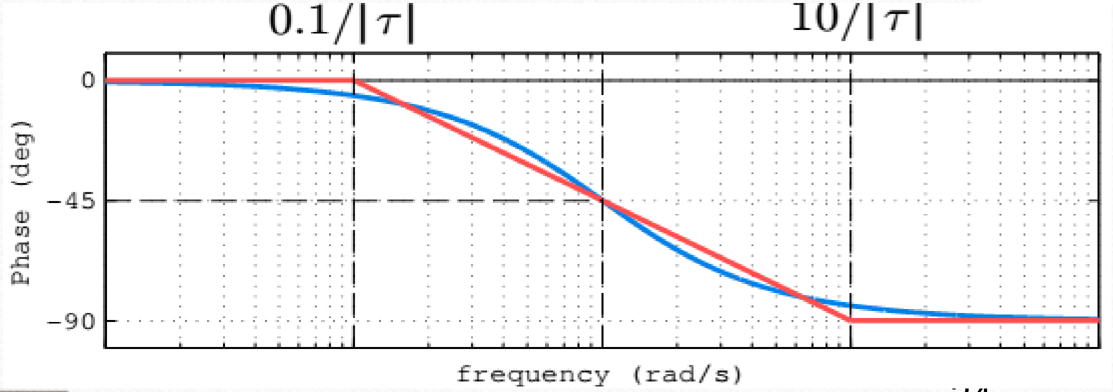
3. Binomial 1/(1 + jωτ) $\frac{a}{(s+a)}$ *or* $(\frac{1}{a}s+1)^{-1}$



$\tau < 0$

$\tau > 0$





Poles, Zeros and Bode Plots

Magnitude:

$$\left|(1+j\frac{\omega}{a})^{-1}\right|_{dB} = -20\log\sqrt{1+(\frac{\omega}{a})^2}$$

$$= -10\log[1+(\frac{\omega}{a})^2]$$

$$\omega \prec\prec a \Rightarrow \frac{\omega}{a} \approx 0 \Rightarrow dB = -10\log 1 = 0$$

$$\omega \succ\succ a \Rightarrow 1+j\frac{\omega}{a} \approx \frac{\omega}{a} \Rightarrow dB \approx -20\log\frac{\omega}{a}$$

$$dB = -[20\log\omega - 20\log a]$$

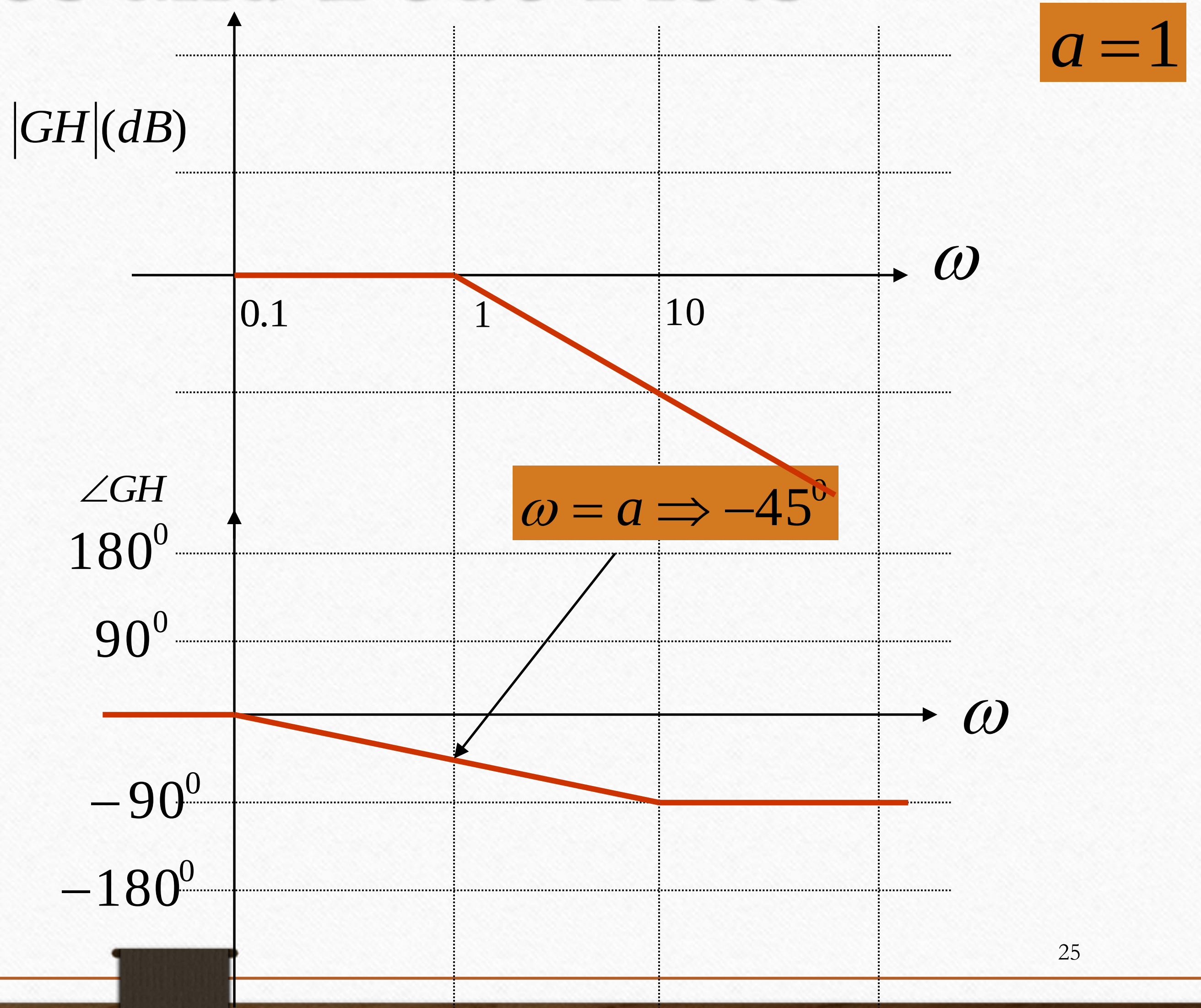
$$\omega = a \Rightarrow 1+j1 \Rightarrow dB = -10\log 2 = -3.01$$

Phase:

$$\angle(1+j\frac{\omega}{a}) = 0^0 - \tan^{-1}\frac{\omega}{a}$$

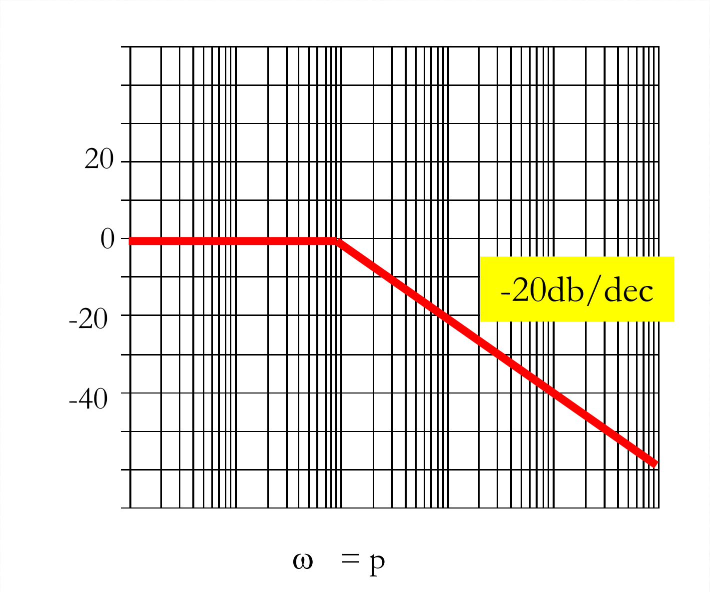
$$\omega \prec\prec a \Rightarrow \frac{\omega}{a} \approx 0 \Rightarrow \angle GH \approx \tan^{-1} 0 = 0^o$$

$$\omega \succ\succ a \Rightarrow \frac{\omega}{a} \approx \infty \Rightarrow \angle GH \approx -\tan^{-1}\infty = -90^o$$



Poles, Zeros and Bode Plots

The term, - 20log|(jw/p + 1), is drawn with the following approximation:
If **w < p** we use the approximation that –20log|(jw/p + 1)| = 0 dB, a flat line on the Bode.
If **w > p** we use the approximation of –20log(w/p), which slopes at -20dB/dec starting at w = p.
It is easy to show that the plot has an error of
-3dB at w = p and – 1 dB at w = p/2 and w = 2p.
One can easily make these corrections if it is appropriate.



Poles, Zeros and Bode Plots

3. Trinomial $1 + 2\zeta(j\omega)/\ \omega_n + (j\omega)^2/\ \omega_n^{\ 2}$

Magnitude

$$\left|1 + 2\frac{\zeta}{\omega_n}(j\omega) + \frac{(j\omega)^2}{\omega_n^2}\right| = \left|1 - \frac{\omega^2}{\omega_n^2} + j2\zeta\frac{\omega}{\omega_n}\right| = \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(4\zeta^2\frac{\omega^2}{\omega_n^2}\right)}$$

$$|\text{TRINOMIAL}|_{dB} \approx \begin{cases} 0\,dB & \text{if} \quad \omega \ll \omega_n \\ 40\log_{10}\omega - 20\log_{10}\omega_n^2 & \text{if} \quad \omega \gg \omega_n \end{cases}$$

Poles, Zeros and Bode Plots

3. Trinomial 1 + 2ζ(jω)/ ω_n + (jω)^2/ $\omega_n^{\ 2}$

Phase

$$T(j\omega)=\frac{\omega_n^2}{(\omega_n^{\ 2}-\omega^2)+2j\xi\omega_n\omega} \qquad \angle T(j\omega)=-\tan^{-1}\frac{2\xi\omega\omega_n}{(\omega_n^{\ 2}-\omega^2)}$$

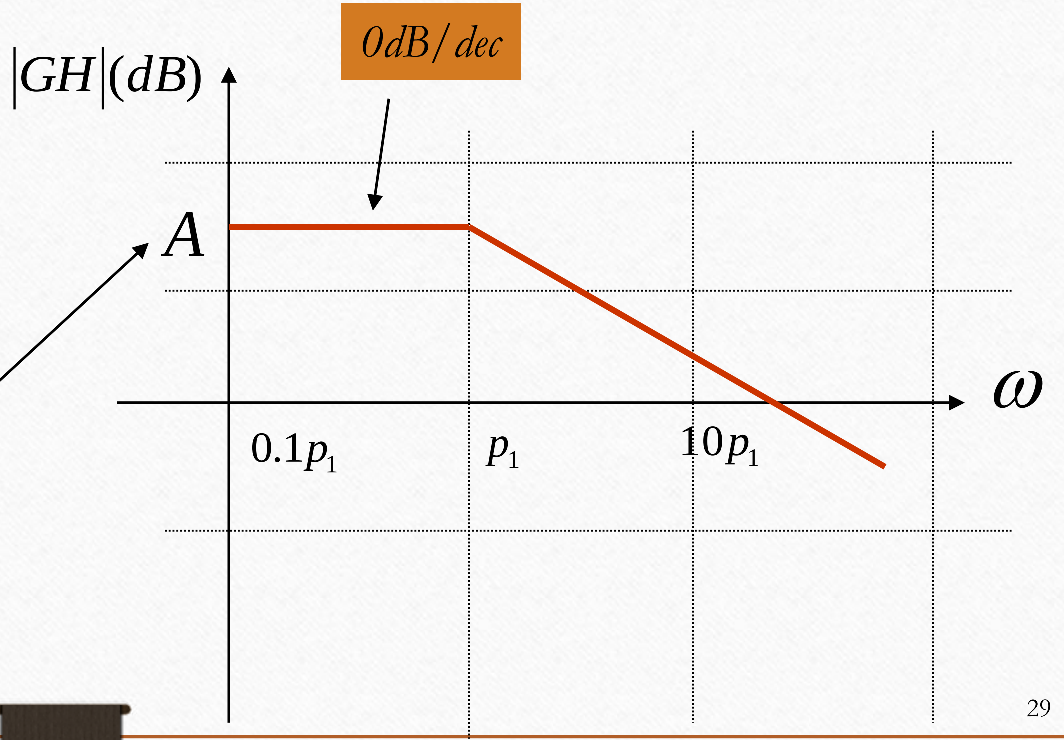
$$T(j\omega)=\frac{1}{(1-(\frac{\omega}{\omega_n})^2)+j2\xi\frac{\omega}{\omega_n}} \qquad \angle T(j\omega)=-\tan^{-1}\frac{2\xi\frac{\omega}{\omega_n}}{1-(\frac{\omega}{\omega_n})^2}$$

Bode Plots

Type 0 : (i.e. n=0)

$$T(s) = \frac{k_p p_1}{(s + p_1)}$$

$20\log K_p = A$



Bode Plots

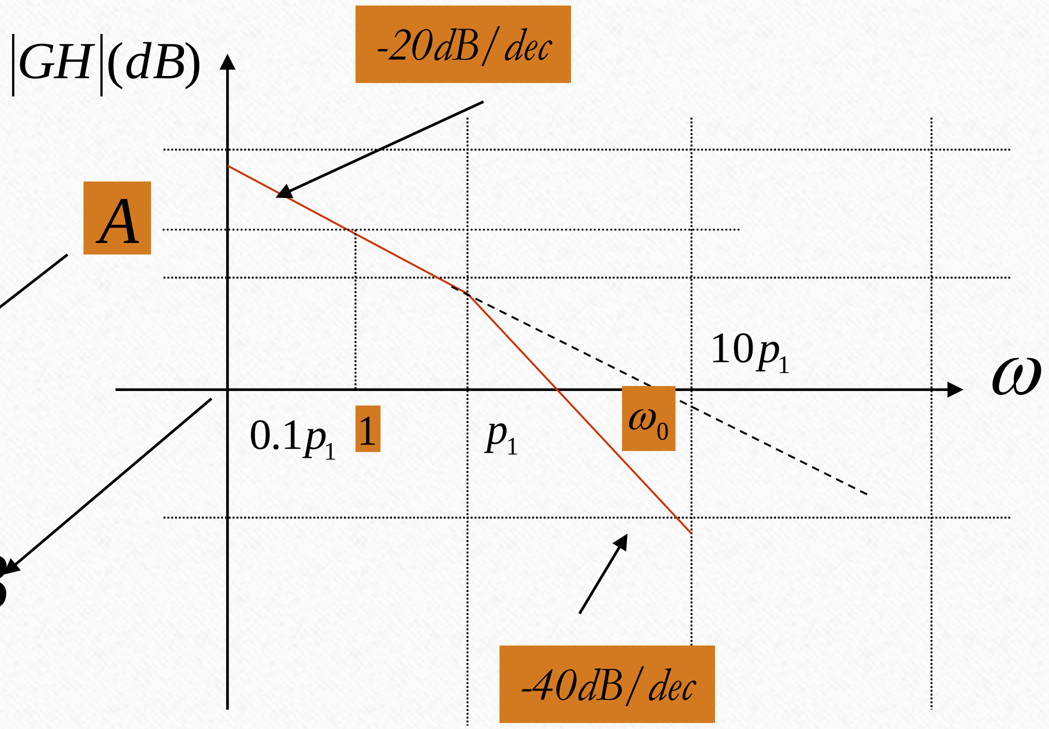
Type I : (i.e. n=1)

$$T(s) = \frac{k_v p_1}{s(s+p_1)}$$

$$20\log K_v = A$$

$$\because \omega_0 = k_v$$

$$20\log \frac{K_v}{j\omega_0} = 0dB$$



Bode Plots

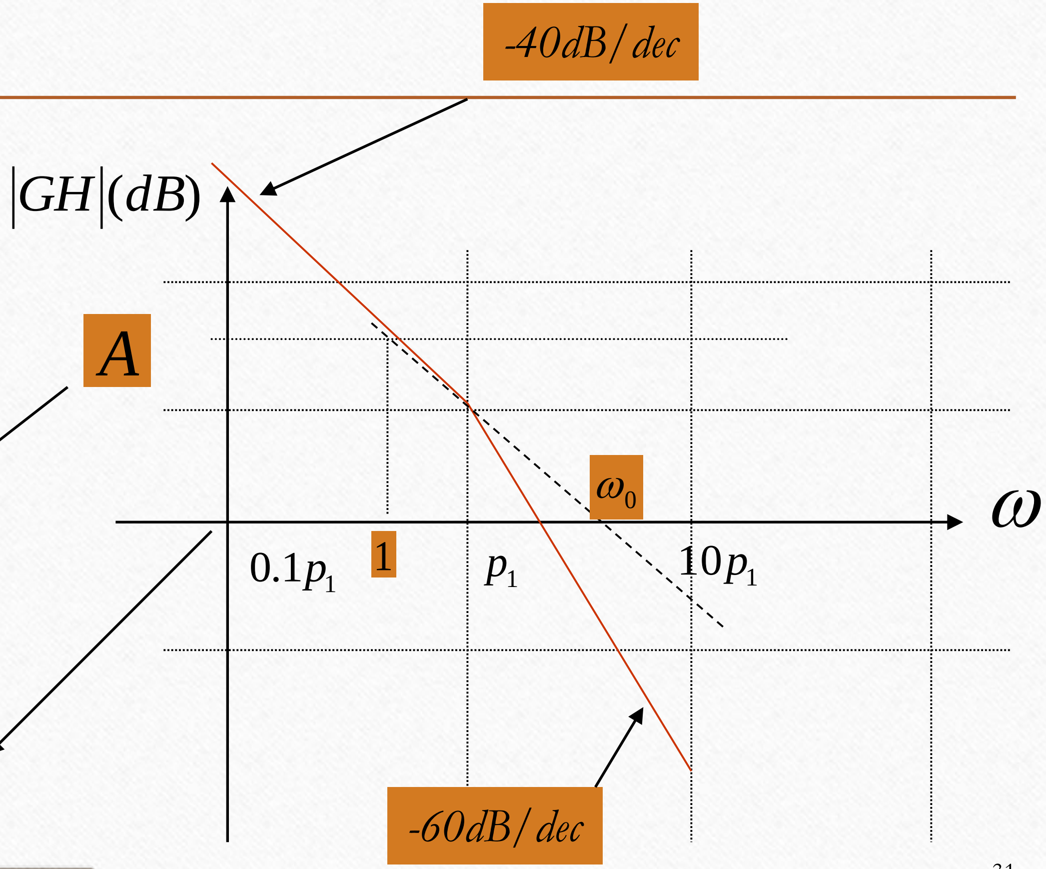
Type 2 : (i.e. n=2)

$$T(s) = \frac{k_a p_1}{s^2(s+p_1)}$$

$$20\log K_a = A$$

$$\because \omega_0^{\,2} = k_a$$

$$20\log \frac{K_a}{(j\omega_0)^2} = 0dB$$



Bode Plots

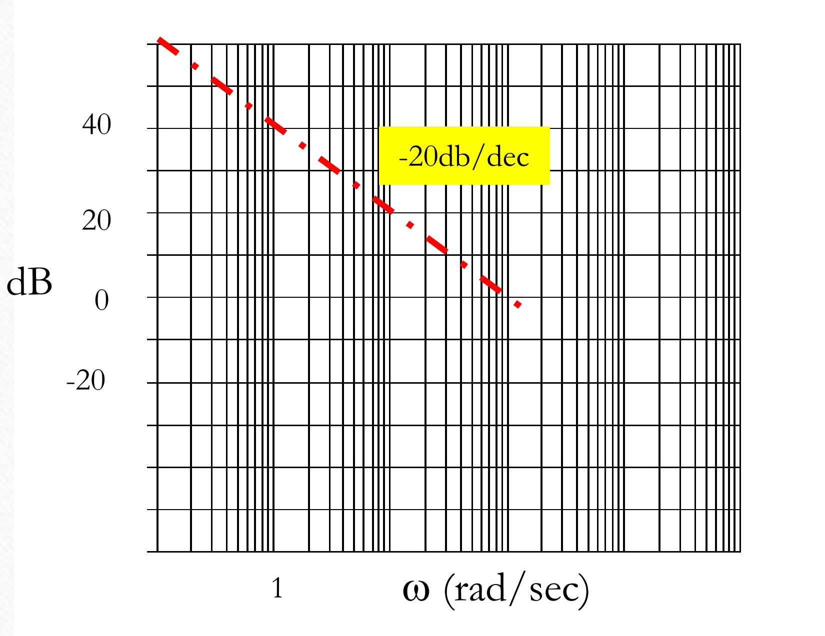
Example : Given the transfer function. Plot the Bode magnitude.

$$G(s) = \frac{100(1+s/10)}{s(1+s/100)^2}$$

Consider first only the two terms of $\frac{100}{jw}$

Which, when expressed in dB, are; 20log100 – 20 logw.
This is plotted below.

The .... is a tentative line we use until we encounter the first pole(s) or zero(s) not at the origin.

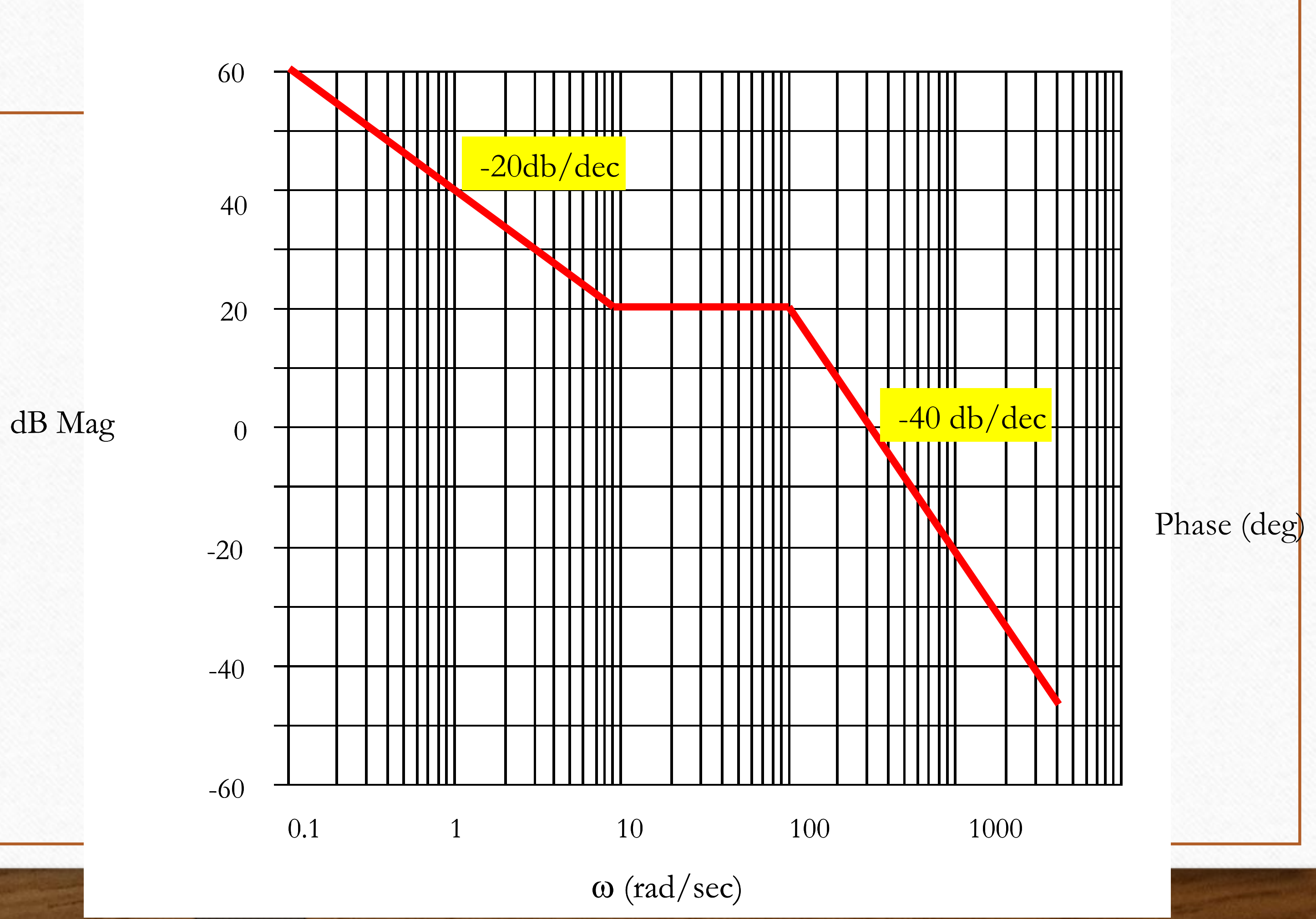


Bode Plots

Example :

The completed plot is shown below.

$$G(s) = \frac{100(1 + s/10)}{s(1 + s/100)^2}$$



Phase

$$\angle G(jw) = \tan^{-1}(w/10) - \tan^{-1}(\infty) - \tan^{-1}(w/100)$$

Model Examples

- Pulse Width Modulation (PWM)